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On a point source in a rotating fluid

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Summary

When a point source of (weak) strength ε is placed in a rotating fluid, Barua [5] and Squire [6] described the local effects which exist in a domain of size $O(\varepsilon^{1/3})$ about the source. Here we show (a) how this can be joined with the linear solution of Moore and Saffman [8] at distances larger than $O(\varepsilon E^{-1})$ from the source (E is the Ekman number), and (b) that when the source is placed between two parallel discs, a vortex develops with its axis through the source.

1. Introduction

There have been many studies of source-sink flows in a rotating fluid. Hide [1], in both a theoretical and experimental study, first considered the effects caused by distributing sources and sinks on cylinder walls in a quite general manner. Many results were developed, but the main conclusion can be summarised as follows: when the sources are on one finite circular cylinder, and the sinks on a second, coaxial cylinder, then the mass flux between the cylinders is through the Ekman layers on the discs; within the interior there is constant circulation. Stewartson layers along the cylinder walls are required to transmit the fluid to and from the Ekman layers. A similar conclusion was found by Barcilon [2] through a linear analysis. Hide extended his discussion to include weakly nonlinear situations, and these have since been made more precise by Bennetts and Hocking [3]. When asymmetries are present further adjustments are necessary, but Greenspan [4] showed that the basic mechanism recounted above is unchanged.

In this paper we consider the fundamental solution corresponding to a source of fluid with (weak) strength ε placed in a rotating fluid.

Some time ago Barua [5] and Squire [6] considered the local effects, close to a source placed on the axis of rotation. They showed how the outflow is restricted within a cylindrical column (a Taylor column) along the axis of rotation, and made attempts to calculate its radius. Because it was treated as a local flow, the problem as posed was indeterminate, and an extra assumption had to be made to give a definite solution. Although the assumptions they took were different, the qualitative conclusions these produced were the same, and it was only in the arithmetic value rather than the order of magnitude of the column radius that they differed. In the first half we observe how the far field behaviour merges with their solution, which is appropriate a distance $O(\varepsilon^{1/3})$ from the source. At distances greater than $O(\varepsilon E^{-1})$ the motion is linear, and similar to that described by Pao and Kao [7], who also showed how to calculate the non-linear perturbations to this linear flow at the edge of the column. In the second part we take the source to be placed between parallel discs; our major conclusion is that a vortex with strength $O(\varepsilon E^{-1/2})$, where E is the Ekman number, develops with its axis through the source. This conclusion is unchanged when the source is more generally placed, and it can be seen, in fact, that vortices are a basic consequence of source-sink flows in a contained rotating fluid. It is the response of the fluid to the mass flux through the Ekman layers which leads to its creation.

2. The fundamental solution

A fluid rotates with constant angular velocity Ω, and we consider the flow due to a source placed at the point O on its vertical axis. Let a be a reference length, and ar, az represent radial and axial distances from the origin O placed on the axis of rotation. When $\Omega au(r, z)$, $\Omega av(r, z)$ and $\Omega aw(r, z)$ are the radial, azimuthal and axial velocities respectively, the equations of motion for axially symmetric flow are

$$u_r + \frac{1}{r}u + w_z = 0, \tag{2.1}$$

$$uu_r + wu_z - \frac{1}{r}v^2 = -p_r + E\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + u_{zz}\right), \tag{2.2}$$

$$uv_r + wv_z + \frac{1}{r}uv = E\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v + v_{zz}\right), \tag{2.3}$$

$$uw_r + ww_z = -p_z + E\left(w_{rr} + \frac{1}{r}w_r + w_{zz}\right), \tag{2.4}$$

where $\rho\Omega^2 a^2 p(r, z)$ is the pressure of the fluid with density ρ; the parameter E is the Ekman number, defined as $\nu/\Omega a^2$, where ν is the kinematic viscosity of the fluid.

When the velocities are small enough for the linear equations to be valid (a statement which will be given greater precision later on) the problem is of classical form and we can, as usual, write

$$u = -\frac{\varepsilon}{r}\psi_z, \quad v = r + \frac{\varepsilon}{r}\chi, \quad w = \frac{\varepsilon}{r}\psi_r, \tag{2.5}$$

where ε is the small Rossby number, to give the governing equations

$$-2\psi_z = E\left(\chi_{rr} - \frac{1}{r}\chi_r + \chi_{zz}\right), \tag{2.6}$$

$$2\chi_z = E\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \psi. \tag{2.7}$$

Following the ideas for the unbounded flow situation discussed by Moore and Saffman [8] for solutions singular at $r = 1$, $z = 0$, it is seen, because there is no Ekman layer, that the z-derivatives in the viscous stresses are negligible. Hence, the general solution to these equations (2.6), (2.7) which are bounded as $z \to \infty$ can be written as

$$\psi = r\int_0^\infty A(k)J_1(kr)\,\mathrm{e}^{-1/2Ek^3 z}\,\mathrm{d}k, \quad \chi = -r\int_0^\infty kA(k)J_1(kr)\,\mathrm{e}^{-1/2Ek^3 z}\,\mathrm{d}k$$

for $z \geqslant 0$, where $A(k)$ is an unknown function; a corresponding expression exists for $z < 0$, but because of the symmetry we restrict the solution to positive z in all which follows. Now there is no length scale for the problem as presently posed, and so a similarity variable must exist, and it is clear that this takes the form

$$\lambda = \frac{r}{(Ez)^{1/3}},$$

(cf. Pao and Kao [7]). Further, the mass flux out of the point source of strength ε, in non-dimensional quantities, requires $\int_0^\infty rw\,\mathrm{d}r = \varepsilon$; that is $\psi(\infty, z) = 1$ for all $z \geqslant 0$. For the general solution for $\psi(r, z)$ to satisfy these requirements it is necessary that $A(k) \equiv 1$, so that

$$\psi = r\int_0^\infty J_1(kr)\,\mathrm{e}^{-1/2Ek^3z}\,\mathrm{d}k, \quad \chi = -r\int_0^\infty kJ_1(kr)\,\mathrm{e}^{-1/2Ek^3z}\,\mathrm{d}k, \quad z \geqslant 0. \tag{2.8}$$

Alternatively, we can write $\psi = \Psi(\lambda)$ and $\chi = (Ez)^{-1/3}X(\lambda)$ for

$$\Psi(\lambda) = \lambda\int_0^\infty J_1(\xi\lambda)\,\mathrm{e}^{-1/2\xi^3}\,\mathrm{d}\xi, \quad X(\lambda) = -\lambda\int_0^\infty \xi J_1(\xi\lambda)\,\mathrm{e}^{-1/2\xi^3}\,\mathrm{d}\xi,$$

from which the velocities can be determined as

$$u = \tfrac{1}{3}\varepsilon E(Ez)^{-4/3}\Psi'(\lambda), \quad v = r + \varepsilon(Ez)^{-2/3}\lambda^{-1}X(\lambda), \quad w = \varepsilon(Ez)^{-2/3}\lambda^{-1}\Psi'(\lambda)$$

to complete the linear solution.

We now test for the introduction of inertial forces. To begin, we rewrite the angular momentum equation (2.3) in the form

$$-2\psi_z + \varepsilon r^{-1}(\psi_r\chi_z - \psi_z\chi_r) = E(\chi_{rr} - r^{-1}\chi_r + \chi_{zz}),$$

using (2.5). When $\lambda = O(1)$, it follows that the inertial terms here are $O\{\varepsilon^2E(Ez)^{-7/3}\}$ while the Coriolis and viscous terms are $O\{\varepsilon E(Ez)^{-4/3}\}$. The non-linear terms must therefore be included once $z = O(\varepsilon E^{-1})$ as z decreases, which implies $r = O(\varepsilon^{1/3})$. (We note that implicit in these calculations is the assumption $|z| \gg |r|$, which requires $\varepsilon \gg E^{3/2}$. A slightly different discussion is necessary for smaller ε, but this is of limited interest, and is not pursued here.) It follows that the inertial forces are present for all ε in some domain close to the source, whose size depends on the relative magnitude of ε and E. A similar calculation shows that inertial and viscous terms have equal magnitude in the axial momentum equation (2.4) when $r = O(\varepsilon^{1/3})$, $z = O(\varepsilon E^{-1})$; the basic orders of magnitude are $u = O(E\varepsilon^{-1/3})$, $v = O(\varepsilon^{1/3})$, $w = O(\varepsilon^{1/3})$, $p = O(\varepsilon^{2/3})$. Clearly the similarity variable λ is appropriate here only in the limit as $Ez/\varepsilon \to \infty$.

There must be a further, distinct domain present where the radial momentum equation includes the inertial terms; scaling arguments show this to be where both r and z are $O(\varepsilon^{1/3})$. In this smallest domain all velocities are $O(\varepsilon^{1/3})$, and the governing equations are just (2.1)–(2.4) without the viscous terms. This is the domain considered by Barua [5], where inertial effects alone dominate. The Taylor column along the z-axis described by Barua is the main feature where r and z are $O(\varepsilon^{1/3})$; the maximum radius at $z = 0$ was given by him as $1.064\varepsilon^{1/3}$, and as $z/\varepsilon^{1/3}$ increases this tends to $0.634\varepsilon^{1/3}$, with a constant axial velocity $0.396\varepsilon^{1/3}$. The basic balance for this essentially inviscid flow is between Coriolis and inertial forces, with viscosity necessary only to remove the discontinuity across the surface of the column. From scaling arguments it is seen that the width of the viscous layer on the discontinuity surface is $O(E^{1/2})$, which is small compared to $O(\varepsilon^{1/3})$ when $\varepsilon \gg E^{3/2}$. The layer is of Blasius type, with a balance between viscous and inertial forces to remove the discontinuity for both the axial and azimuthal velocities. The similarity variable for this Blasius flow is $\varepsilon^{1/6}E^{-1/2}(r-\varepsilon^{1/3}\lambda)z^{-1/2}$ where $r=\varepsilon^{1/3}\lambda(z)$ is the equation of the discontinuity surface. Therefore, when $z=O(\varepsilon E^{-1})$ this layer occupies the whole column where $r=O(\varepsilon^{1/3})$, and there is then a general balance between viscous, inertial and Coriolis forces throughout the column.

The values for the radius of the Taylor column could only be gained by Barua after making certain assumptions regarding the flow outside the column, and then invoking a variational principle. There is an indeterminacy in considering the behaviour within the r, $z=O(\varepsilon^{1/3})$ domain as a local flow which can only be resolved on such a basis. Squire [6], made an alternate set of assumptions, and although the qualitative description of the flow as recounted above is unchanged, he found the maximum radius to be $0.939\ \varepsilon^{1/3}$, and the asymptotic value for the radius as $z/\varepsilon^{1/3}$ increases tends to $0.766\ \varepsilon^{1/3}$, where the constant axial velocity is $0.542\ \varepsilon^{1/3}$, the difference from the values Barua calculated are of the order of 25%; however, these differences are irrelevant for the subsequent discussion.

We can see now that (2.8) represents the far field solution when viscous and Coriolis forces alone balance, the inertial forces have weakened at points a distance greater than $O(\varepsilon E^{-1})$ from the source to the extent that they can be neglected.

The second part of the present discussion is now to more naturally place the source between two discs. For mathematical simplicity, we place the source midway between the discs without restricting the main conclusion. To be specific, we consider the motion due to a source of strength ε placed at the origin between discs which are now placed along $z=\pm d$; the whole system still rotates with angular velocity Ω.

We again solve the equations (2.6), (2.7) for points away from the Ekman layers which exist along each of the discs, through neglecting the z-derivatives in the viscous stresses. Because the z-domain is now bounded, extra terms will be present, though these can be reduced to just one by invoking symmetry in z. It follows that the general solution with the required symmetry can be expressed as

$$u=\tfrac{1}{2}\varepsilon E\int_0^\infty\left\{k^3\,\mathrm{e}^{-1/2Ek^3z}-k^2C(k)\cosh\left(\tfrac{1}{2}Ek^3z\right)\right\}\mathrm{d}k,$$

$$v=r-\varepsilon\int_0^\infty\left\{k\,\mathrm{e}^{-1/2Ek^3z}+C(k)\cosh\left(\tfrac{1}{2}Ek^3z\right)\right\}\mathrm{d}k,$$

$$w=\varepsilon\int_0^\infty\left\{k\,\mathrm{e}^{-1/2Ek^3z}+C(k)\sinh\left(\tfrac{1}{2}Ek^3z\right)\right\}\mathrm{d}k,$$

$$p=\tfrac{1}{2}r^2+2\varepsilon\int_0^\infty\left\{\mathrm{e}^{-1/2Ek^3z}-k^{-1}C(k)\cosh\left(\tfrac{1}{2}Ek^3z\right)\right\}\mathrm{d}k,$$

for $z\geqslant 0$; $C(k)$ is an unknown function. The first term in each of the integrals (2.9) represents the source, and the second term is the correction due to the presence of the discs. The integrals are valid for all points of the fluid except those within the Ekman layers along the discs. However, to satisfy the conditions on the discs we can use the Ekman compatability condition; the width of the columnar region along the axis is certainly large compared to the Ekman layer thickness $O(E^{1/2})$. The Ekman condition can be expressed in the form

$$w = -\tfrac{1}{4}E^{1/2}\left(\bar{p}_{rr} + \frac{1}{r}\bar{p}_r\right) \quad \text{on} \quad z = d, \tag{2.10}$$

when $\bar{p} = p - \frac{1}{2}r^2$. Substitution of the integrals (2.9) into (2.10) shows

$$C(k) \simeq \frac{\left(\frac{1}{2}E^{1/2}k^2 - k\right)\mathrm{e}^{-1/2Ek^3d}}{\sinh\left(\frac{1}{2}Ek^3d\right) + \frac{1}{2}E^{1/2}k\cosh\left(\frac{1}{2}Ek^3d\right)}. \tag{2.11}$$

With the denominator of (2.11) having the familiar form of Stewartson layer analyses, it is clear that the columnar region about the z-axis has cylindrical domains with radii $O(E^{1/3})$ and $O(E^{1/4})$. It is the latter, wider domain which is of greater interest, and when we take asymptotic approximations it follows that

$$v - r \simeq 2\varepsilon E^{-3/4}d^{-1}\int_0^\infty \left(\gamma^2 + d^{-1}\right)^{-1} J_1(\gamma\rho)\,\mathrm{d}\gamma$$

when $r = E^{1/4}\rho$ with $\rho = O(1)$. This integral can be evaluated (c.f. Erdelyi et al. [9]) for

$$v - r \simeq 2\varepsilon E^{-3/4}\left\{\rho^{-1} - d^{-1/2}K_1\left(d^{-1/2}\rho\right)\right\}. \tag{2.12}$$

Therefore, outside the column of width $r = O(E^{1/4})$, we can write

$$v = r + 2\varepsilon E^{-1/2}r^{-1}$$

in the body of the fluid to show the presence of a vortex with strength $O(\varepsilon E^{-1/2})$. Physically, the fluid created at the source is transmitted vertically within the column with width $O(E^{1/3})$, and then dispersed within the Ekman layers. These are divergent layers, and the radial velocity there is proportional to r^{-1}; this, in turn, induces an azimuthal velocity proportional to r^{-1}, the potential vortex. The axial velocity and pressure corresponding to (2.12) are

$$w \simeq \varepsilon E^{-3/4}d^{-2}|z|K_0\left(d^{-1/2}\rho\right), \quad p = \tfrac{1}{2}r^2 + 2\varepsilon E^{-1/2}\left\{\ln\rho - K_0\left(d^{-1/2}\rho\right)\right\}. \tag{2.13}$$

Two final comments. If the source is placed at a more general point $(r_0, \theta_0, 0)$ off the axis in a fluid of infinite extent, the governing equations in a co-ordinate system rotating with the fluid become

$$-4W_z = E\nabla^4 P, \quad P_z = E\nabla^2 W$$

following Greenspan [4]; here the variables P and W refer to the pressure and axial velocity in the rotating frame. The constraints of the Taylor-Proudman theorem still ensure that the flow is columnar in nature, and so the z-derivatives in the Laplacian are negligible. Consequently, $\nabla^2 \simeq \partial^2/\partial R^2 + R^{-1}\partial/\partial R$, where R is the radial distance in the polar co-ordinate system centred at $(r_0, \theta_0, 0)$, and the local behaviour is independent of its position in the fluid. The logarithmic singularity for the pressure given in (2.13) is still retained with only the change in co-ordinates; the singularity was shown to be indicative of a vortex in a similar situation by Kranenberg [10].

Secondly, we note that the potential vortex, a consequence of the presence of the mass source, is not present when other singularities exist in the fluid. For example, the calculations for a point source of angular momentum with strength ε can be made parallel to those which led to (2.8) and to (2.12). Without giving details, it can be shown that

$$\chi = \tfrac{1}{8}Er\int_0^\infty k^3 J_1(kr)\,\mathrm{e}^{-1/2Ek^3z}\,\mathrm{d}k, \quad \text{corresponding to (2.8)},$$

and

$$v - r \simeq \tfrac{1}{4}\varepsilon E^{-1/4}d^{-1}\int_0^\infty \alpha^2(\alpha^2 + d^{-1})^{-1}J_1(\alpha\rho)\mathrm{d}\alpha, \quad \text{corresponding to (2.12)};$$

this latter integral has exponential decay as $\exp(-d^{-1/2}\rho)$ for $\rho \to \infty$, indicating that the influence of the point source of angular momentum is totally constrained within the column of radius $O(E^{1/4})$. A similar conclusion follows for a dipole in the z-direction at the origin.

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